3. Eigenvalues

The eigenvalues for the following six cases of homogeneous boundary conditions are investigated:

Case (*i*) $H_e = 0, \quad M_{\phi e} = 0$ (14)

The characteristic equation takes the form

$$S_1S_4 - S_2S_3 = 0 \tag{15}$$

The left-hand side of Eq. (15) is equal to Δ. The influence coefficients thus become indefinitely large at the value of the pressure satisfying Eq. (15). Therefore, it is concluded that the singularity in the influence coefficients corresponds to the eigenvalue and, consequently, to the buckling pressure of the spherical shell segment supported by the edge condition characterized by Eqs. (14).

The stiffness coefficients $K_{ij}, i, j = 1, 2$, are

$$K_{ij} = \bar{C}_{ij}/(C_{11}C_{22} - C_{12}C_{21}) \tag{16}$$

where $\bar{C}_{ij}$ is the co-factor of C_{ij} in the determinant $|C_{ij}|$.

A simple calculation shows

$$C_{11}C_{22} - C_{12}C_{21} = (\lambda_0^6 \sin^2\alpha/E^2h^2)(1/\Delta)[(\gamma_1S_2' - \gamma_2S_1') - \nu \cot\alpha(\gamma_1S_2 - \gamma_2S_1)] \tag{17}$$

Hence, Δ cancels out and K_{ij} is not necessarily zero when $\Delta = 0$. The result is not surprising at all, because the buckling deformations occur in the eigenmode, and, consequently, u_e and β_e are linearly related and their ratio cannot be changed arbitrarily. Therefore, the stiffness coefficients need not vanish identically.

The boundary condition characterized by Eqs. (14) is quite unrealistic except for hemispherical shells. As sketched in Fig. 1, the shell slides and rotates freely on the constraining surface of conical shape, which becomes horizontal when the buckling occurs. In the case of the hemispherical shell the constraining surface is always horizontal. The eigenvalue $\rho = 0.5$ obtained for the hemispherical shell, therefore, corresponds to the buckling pressure of the hemispherical shell with free edge. Because of the similarity of their behavior near the edge, it can be anticipated that the axially compressed circular cylindrical shell with free edge can buckle at one-half the classical buckling pressure. This is the problem discussed by N. J. Hoff.[6]

Case (*ii*) $Q_{\alpha e} = 0, \quad M_{\phi e} = 0$ (18)

where $Q_{\alpha e}$ is the component of the edge force in the radial direction specified by the edge angle α.

Approximation consistent to Eq. (10) results in

Case $Q_{\alpha e} = - H_e \sin\alpha$ (19)

Therefore, in the present approximation, the boundary condition characterized by Eqs. (18) is identical to that characterized by Eqs. (14). The present eigenvalue problem is a realistic one. Here, the shell slides and rotates freely on the constraining surface of conical shape during the entire process of loading and buckling, Fig. 2.

Because of the similarity of their behavior near the edge, the axisymmetric buckling of conical shells, which can slide and rotate freely on the constraining surface of conical shape, can be anticipated to occur at the pressure corresponding to the eigenvalue of the present boundary value problem. The corresponding conical shell is sketched by dashed lines in Fig. 2. This conclusion may provide a supplementary explanation to the result of the recent investigation of Baruch, Harari, Singer.[4] They obtained the low buckling load of the conical shell near $\rho = 0.5$ for the SS3 boundary condition, which turned out to be similar to the free edge for that buckling mode.

$$\begin{aligned} &\text{Case } (iii) \quad Q_e = 0, \quad M_{\phi e} = 0 \\ &\text{Case } (iv) \quad H_e = 0, \quad \beta_e = 0 \\ &\text{Case } (v) \quad u_e = 0, \quad M_{\phi e} = 0 \\ &\text{Case } (vi) \quad u_e = 0, \quad \beta_e = 0 \end{aligned} \tag{20}$$

The result of numerical computations showed that $\rho = 1$ is the only eigenvalue for all these cases of boundary conditions.

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A Direct Modification Procedure for the Displacement Method

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Introduction

THE increased speed and capacity of modern computers is permitting the solution of large nonlinear or optimization problems using the Matrix Displacement Method. An efficient modification procedure is an integral feature of the required computer programs. When a relatively small portion of the structure is to be modified at one time, or when convergence difficulties are expected in an iterative approach, a direct method[1–5] can be recommended. In the original Argyris approach,[1,2] the changes in values of the elemental stiffness matrix **k** are interpreted in terms of initial stresses or loads. This method normally involves the triangularization of a matrix of size equal to the number of changed rows in **k** according to the natural element freedoms, and has special application for the elastoplastic problem, as shown in Ref. 3. Sobieszczanski[4] has presented a competitive method for the case of successive independent modifications to a structure. Since the method requires element flexibility matrices, its use in conjunction with existing Displacement Method programs would involve considerable programing effort. Also, this technique is not especially suited to the elastoplastic problem, where all the previously modified members must be modified anew after each load increment.

The size of the modifications matrices can often be considerably reduced if changes are made directly with respect to the global stiffness matrix **K** of the structure. Earlier attempts in this direction presumed that **K** was partitioned into modified and unmodified parts. Even if this transformation

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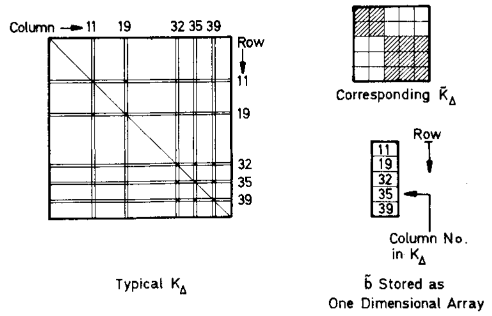

Fig. 1 Formation of compressed modification matrix.

were only done implicitly, the solution of large out-of-core problems would involve excessive input-output time. Also, the recent method of Sack, Carpenter, and Hatch[5] is not well adapted to modifications of large structures because 1) it makes the modifications directly to $\mathbf{K}^{-1}$, which is extremely expensive to calculate for large band matrices, 2) it can only consider the modification of one column of $\mathbf{K}$ at a time, 3) the unsymmetrical form of the transformation $\mathbf{t}^t(\mathbf{K}^{-1})_i\mathbf{u}$ in Eq. (3) of Ref. 5 necessitates both forward and back substitution on $\mathbf{u}$, the latter being very expensive. In the following presentation, these disadvantages are overcome to produce directly the correction in displacements for a structure, where without prior knowledge, any members are added, deleted or changed.

Derivation from Matrix Calculus

Consider that the displacements $\mathbf{r}$ are calculated for a structure with global stiffness matrix $\mathbf{K}$ under loading system $\mathbf{R}$ as

$$\mathbf{K}\mathbf{r} = \mathbf{R} \tag{1}$$

The change in displacements $\mathbf{r}_\Delta$ caused by a change in $\mathbf{K}$ of $\mathbf{K}_\Delta$ is expressed through

$$(\mathbf{K} + \mathbf{K}_\Delta)(\mathbf{r} + \mathbf{r}_\Delta) = \mathbf{R} \tag{2}$$

As illustrated in Fig. 1, the incremental matrix $\mathbf{K}_\Delta$ could be compressed, eliminating zero columns and rows, to form a reduced incremental matrix $\tilde{\mathbf{K}}_\Delta$ of size equal to the number of changed columns (or rows) in the modified system, via

$$\mathbf{K}_\Delta = \tilde{\mathbf{b}}^t\tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}} \tag{3}$$

where $\tilde{\mathbf{b}}$ is a Boolean matrix with linearly independent rows, each of which contains all zeros except for one unit value, located for each row at the column number, taken in increasing order, where a change in $\mathbf{K}$ occurs. Note the orthonormal condition

$$\tilde{\mathbf{b}}^t\tilde{\mathbf{b}} = \mathbf{I} \tag{4}$$

However, this relation is strictly not necessary for the subsequent developments. It suffices to remember that the rows of $\tilde{\mathbf{b}}$ are linearly independent. The structure of $\tilde{\mathbf{b}}$ enables it to be stored as a one-dimensional array, as seen in Fig. 1.

The identity of Householder[7] for a modified inverse is given as

$$(\mathbf{A} + \mathbf{X}^t\mathbf{Y})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{X}^t(\mathbf{I} + \mathbf{Y}\mathbf{A}^{-1}\mathbf{X}^t)^{-1}\mathbf{Y}\mathbf{A}^{-1} \tag{5}$$

where $(\mathbf{I} + \mathbf{Y}\mathbf{A}^{-1}\mathbf{X}^t)$ can be proved nonsingular if both the original matrix $\mathbf{A}$ and the modified matrix $(\mathbf{A} + \mathbf{X}^t\mathbf{Y})$ are taken to be nonsingular. In the structural problem, set $\mathbf{A} = \mathbf{K}$, $\mathbf{X} = \tilde{\mathbf{b}}$ and $\mathbf{Y} = \tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}}$ in Eq. (5) to give

$$(\mathbf{K} + \tilde{\mathbf{b}}^t\tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}})^{-1} = \mathbf{K}^{-1} - \mathbf{K}^{-1}\tilde{\mathbf{b}}^t(\mathbf{I} + \tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}}\mathbf{K}^{-1}\tilde{\mathbf{b}}^t)^{-1}\tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}}\mathbf{K}^{-1} \tag{6}$$

Since the calculation of the inverse is too expensive for a large band matrix, Eq. (6) must be multiplied through by $\mathbf{R}$, and with the help of Eq. (2), the change in displacements $\mathbf{r}_\Delta$ obtained directly as

$$\mathbf{r}_\Delta = -\mathbf{K}^{-1}\tilde{\mathbf{b}}^t(\mathbf{I} + \tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}}\mathbf{K}^{-1}\tilde{\mathbf{b}}^t)^{-1}\tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}}\mathbf{r} \tag{7}$$

The reduced unsymmetric matrix $(\mathbf{I} + \tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}}\mathbf{K}^{-1}\tilde{\mathbf{b}}^t)$ can be readily shown to be nonsingular, even when $\tilde{\mathbf{K}}_\Delta$ is singular.

Derivation from Physical Arguments

Due to the change $\mathbf{K}_\Delta$ of the stiffness matrix, the displacements $\mathbf{r}$ of the unmodified system under a given load $\mathbf{R}$ are incremented by $\mathbf{r}_\Delta$. This displacement vector $\mathbf{r}_\Delta$ may be derived from additional forces

$$\mathbf{R}_\Delta = \mathbf{K}\mathbf{r}_\Delta \tag{8}$$

applied to the unmodified system. For the load vector $\mathbf{R}_\Delta$ we deduce the condition

$$\mathbf{K}_\Delta\mathbf{r} + \mathbf{R}_\Delta + \mathbf{K}_\Delta\mathbf{K}^{-1}\mathbf{R}_\Delta = \mathbf{0} \tag{9}$$

by eliminating $\mathbf{r}_\Delta$ in Eq. (2) via Eq. (8). We observe that $\mathbf{R}_\Delta$ is uniquely determined by Eq. (9). However, the order of the system of linear equations may be most elegantly reduced. To this purpose we set

$$\mathbf{R}_\Delta = \tilde{\mathbf{b}}^t\tilde{\mathbf{R}}_\Delta \tag{10}$$

where $\tilde{\mathbf{b}}$ is the Boolean matrix defined in connection with Eq. (3).

This means physically that additional forces are only applied where the corresponding rows of the stiffness matrix are changed. Introducing Eqs. (10) and (3) into Eq. (9) we find

$$\tilde{\mathbf{b}}^t(\tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}}\mathbf{r} + \tilde{\mathbf{R}}_\Delta + \tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}}\mathbf{K}^{-1}\tilde{\mathbf{b}}^t\tilde{\mathbf{R}}_\Delta) = \mathbf{0} \tag{11}$$

Remembering the above definition of $\tilde{\mathbf{b}}$, it is clear, that for any nonzero $\tilde{\mathbf{R}}_\Delta$, Eq. (10) leads to nonzero $\mathbf{R}_\Delta$. There follows that the left-hand side of Eq. (11) vanishes only if the term enclosed in brackets becomes itself zero. Solving the remaining linear equation for $\tilde{\mathbf{R}}_\Delta$ we obtain

$$\tilde{\mathbf{R}}_\Delta = -(\mathbf{I} + \tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}}\mathbf{K}^{-1}\tilde{\mathbf{b}}^t)^{-1}\tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}}\mathbf{r} \tag{12}$$

In contrast to Eq. (9) in $\mathbf{R}_\Delta$, the number of equations is here equal to the number of modified rows or columns in $\mathbf{K}$. The possibility of reducing Eq. (9) to the form in Eq. (12), can also be deduced from the "structure" of the matrix $\mathbf{K}_\Delta$. Close examination of Eq. (9) shows that full zero rows must occur in $\mathbf{R}_\Delta$ corresponding to those in $\mathbf{K}_\Delta$, which checks with Eq. (10). In conjunction with Eqs. (8) and (10) we finally derive

$$\mathbf{r}_\Delta = \mathbf{K}^{-1}\mathbf{R}_\Delta = \mathbf{K}^{-1}\tilde{\mathbf{b}}^t\tilde{\mathbf{R}}_\Delta = -\mathbf{K}^{-1}\tilde{\mathbf{b}}^t(\mathbf{I} + \tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}}\mathbf{K}^{-1}\tilde{\mathbf{b}}^t)^{-1}\tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}}\mathbf{r} \tag{13}$$

which is seen to be identical to Eq. (7).

Application of Method

The compressed modification matrix $\tilde{\mathbf{K}}_\Delta$ can be directly calculated from the diagonal hypermatrix $\mathbf{k}_\Delta$, which contains in order the matrices of change of elemental stiffness for each modified element, via

$$\tilde{\mathbf{K}}_\Delta = \tilde{\mathbf{a}}^t\mathbf{k}_\Delta\tilde{\mathbf{a}} \tag{14}$$

where the Boolean matrix $\tilde{\mathbf{a}}$ has the same number of rows as the part of the normal Boolean matrix $\mathbf{a}$ for the changed elements. It has as many columns as $\tilde{\mathbf{b}}$ has rows, and can be built from $\mathbf{a}$ one row at a time, by searching $\tilde{\mathbf{b}}$ for a row r_1,

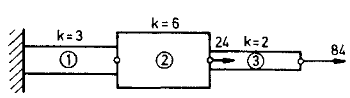

Fig. 2 Example problem—three series bars.

identical to the current row in $\mathbf{a}$, and inserting unity at column r_1 of the current row of $\tilde{\mathbf{a}}$. Assuming that only modified elements are considered in $\mathbf{a}$, the following matrix relation holds

$$\tilde{\mathbf{a}} = \mathbf{a}\tilde{\mathbf{b}}^t \tag{15}$$

giving an alternative method of calculating $\tilde{\mathbf{a}}$.

It is assumed that the original stiffness matrix $\mathbf{K}$ has already been triangularized by the Cholesky method into the upper triangular matrix $\mathbf{u}$ such that

$$\mathbf{u}^t\mathbf{u} = \mathbf{K} \tag{16}$$

Using $\mathbf{u}$, a forward substitution process performed on $\tilde{\mathbf{b}}^t$ (considered as expanded to a standard matrix) produces a rectangular matrix

$$\mathbf{Z} = (\mathbf{u}^t)^{-1}\tilde{\mathbf{b}}^t \tag{17}$$

Finally the symmetric influence matrix of unit changes $\mathbf{Q}$ is defined as

$$\mathbf{Q} = \mathbf{Z}^t\mathbf{Z} \tag{18}$$

Using the above relations, Eq. (7) can be written as

$$\mathbf{r}_\Delta = -\mathbf{u}^{-1}\mathbf{Z}(\mathbf{I} + \tilde{\mathbf{K}}_\Delta\mathbf{Q})^{-1}\tilde{\mathbf{K}}_\Delta\mathbf{b}\mathbf{r} \tag{19}$$

The equation can be put in convenient positive definite form by extracting the matrix $\mathbf{Q}$ as a common factor from the matrix to be inverted, as follows

$$\mathbf{r}_\Delta = -\mathbf{u}^{-1}\mathbf{Z}\mathbf{Q}^{-1}(\mathbf{Q}^{-1} + \tilde{\mathbf{K}}_\Delta)^{-1}\tilde{\mathbf{K}}_\Delta\tilde{\mathbf{b}}\mathbf{r} \tag{20}$$

$\mathbf{Q}$ is clearly positive definite, and since after certain manipulations the matrix $(\mathbf{Q}^{-1} + \tilde{\mathbf{K}}_\Delta)$ can also be shown to be positive definite, pivoting is unnecessary in the triangularization. The optimal order of calculation in Eq. (20) is clearly 1) compress $\mathbf{r}$ into $\tilde{\mathbf{r}} = \tilde{\mathbf{b}}\mathbf{r}$, and premultiply by $\tilde{\mathbf{K}}_\Delta$ to give $\tilde{\mathbf{r}}'$; 2) evaluate matrices $\mathbf{Z}$ and $\mathbf{Q}$; 3) triangularize $\mathbf{Q}$, i.e., $\mathbf{u}_Q{}^t\mathbf{u}_Q = \mathbf{Q}$, also evaluate $\mathbf{Q}^{-1}$; 4) triangularize $(\mathbf{Q}^{-1} + \tilde{\mathbf{K}}_\Delta)$ into $\tilde{\mathbf{u}}^t\tilde{\mathbf{u}}$; 5) using results from 3, forward and back-substitute on $\tilde{\mathbf{r}}'$ using $\tilde{\mathbf{u}}$ and then $\mathbf{u}_Q$, to produce $\tilde{\mathbf{r}}''$; 6) premultiply $\tilde{\mathbf{r}}''$ by $\mathbf{Z}$ to produce $\mathbf{r}'''$; and 7) finally produce $\mathbf{r}_\Delta$ by back-substitution on $\mathbf{r}'''$, i.e., $-\mathbf{r}_\Delta = \mathbf{u}^{-1}\mathbf{r}'''$.

An operation count assuming a banded $\mathbf{K}$ reveals that step 2, namely the evaluation of $\mathbf{Z}$ and $\mathbf{Q}$, is usually the most expensive. However, if only the *magnitude* of a set of modifications changes in a later modification step, matrices $\mathbf{Z}$ and $\mathbf{Q}$ remain the same as before, and the modification becomes very cheap. From Eq. (17), it can be appreciated, that if the modifications occur at the higher numbered nodal points of the structure, the forward-substitution procedure to form $\mathbf{Z}$ and the resulting calculation of $\mathbf{Q}$ are both performed very rapidly, as is also the case for the efficient substructure approach in Ref. 6.

An operation count will now be given for the major part of the calculations. Assuming that the half-bandwidth B of the structural stiffness matrix is considerably smaller than the total number N of unknown displacements, and that the number of loadings $L \ll B$, only steps 2–4 need be considered. Defining the total number of changed columns in $\mathbf{K}$ as n_c and assuming that the changes occur on the average at the middle nodal point numbers, the number of multiply-accumulates n is given approximately by

$$n = NBn_c/2 + Nn_c^2/4 + 2n_c^3/3 \tag{21}$$

As the number of operations for retriangularization equals $NB^2/2$, retriangularization would be cheaper when n_c becomes greater than about $0.75\,B$. Of course, n_c could be increased beyond this if the changes occur at higher numbered nodal points or if only the magnitude of the modifications changes from a previous step. The indicated operations 1–7 are seen to include mainly matrix multiplication and triangularization. Since these features are available in most general purpose structural programs, the implementation of the above approach will not require much additional programing effort.

Space considerations preclude a thorough comparison of operation counts for this method with those for existing methods. However, it may be mentioned that if the method in Ref. 5 were generalized as above, to enable multiple corrections directly to the displacement matrix, the necessity of back-substitution would raise the operation count to

$$n = 3NBn_c/2 + n_c^3 \tag{22}$$

which even at the most favourable limit value of $n_c = 0.75\,B$, implies more than double the operations given in Eq. (21). It is important to emphasize that the method described above, like all *direct* methods presented until now, is applicable to situations where a relatively small proportion of the structure is modified. Minor modifications to a large part of the structure are probably best handled by an iterative solution of the modified equation system, using the original displacement vector as starting vector.

Numerical Example

A simple example of three series bars Fig. 2 is presented to illustrate the formation of the various matrices. For the *original* structure, the various matrices are seen to be

$$\mathbf{k} = \begin{bmatrix} 3 & -3 & & & & \\ -3 & 3 & & & & \\ & & 6 & -6 & & \\ & & -6 & 6 & & \\ & & & & 2 & -2 \\ & & & & -2 & 2 \end{bmatrix}; \quad \mathbf{a} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

Thus

$$\mathbf{K} = \begin{bmatrix} 9 & -6 & 0 \\ -6 & 8 & -2 \\ 0 & -2 & 2 \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{24}$$

and

$$\mathbf{R} = \begin{bmatrix} 0 \\ 24 \\ 84 \end{bmatrix}; \quad \mathbf{r} = \begin{bmatrix} 36 \\ 54 \\ 96 \end{bmatrix} \tag{25}$$

Consider that k_3 is increased by 5 to a value of 7. Then using Eqs. (14) and (15), $\mathbf{k}_\Delta$, $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}}$ are calculated to be

$$\mathbf{k}_\Delta = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}; \quad \tilde{\mathbf{b}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \tilde{\mathbf{a}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{26}$$

From Eqs. (14, 17, and 18), we obtain

$$\tilde{\mathbf{K}}_\Delta = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}; \quad \mathbf{Z} = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}; \quad \mathbf{Q} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{bmatrix} \tag{27}$$

$$\mathbf{u}_Q = \begin{bmatrix} (0.5)^{1/2} & (0.5)^{1/2} \\ 0 & (0.5)^{1/2} \end{bmatrix}; \quad \mathbf{Q}^{-1} = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix};$$

$$Q^{-1} + \tilde{\mathbf{K}}_\Delta = \begin{bmatrix} 9 & -7 \\ -7 & 7 \end{bmatrix} \tag{28}$$

$$\tilde{\mathbf{u}} = \begin{bmatrix} 3 & -2.33 \\ 0 & 0.33(14)^{1/2} \end{bmatrix}; \quad \tilde{\mathbf{r}} = \begin{bmatrix} 54 \\ 96 \end{bmatrix}; \quad \tilde{\mathbf{r}}' = \begin{bmatrix} -210 \\ 210 \end{bmatrix} \tag{29}$$

$$(\tilde{\mathbf{u}}^t\tilde{\mathbf{u}})^{-1}\tilde{\mathbf{r}}' = \begin{bmatrix} 0 \\ 30 \end{bmatrix}; \quad \tilde{\mathbf{r}}'' = \begin{bmatrix} -60 \\ 60 \end{bmatrix}; \quad \mathbf{r}''' = \begin{bmatrix} 0 \\ -30 \\ 30 \end{bmatrix} \tag{30}$$

thus leading finally to

$$\mathbf{r}_\Delta = -\begin{bmatrix} 0 \\ 0 \\ 30 \end{bmatrix} \quad \text{and} \quad \mathbf{r} + \mathbf{r}_\Delta = \begin{bmatrix} 36 \\ 54 \\ 66 \end{bmatrix} \tag{31}$$

It must be emphasized that the preceding example represents an extremely inefficient application of the method. Nevertheless, the accompanying progressive results should enable the reader to obtain a clearer understanding of the approach.

Note that after submitting the original manuscript, the authors received the paper by D. Kavlie and G. H. Powell.[8] The reader may be referred to this paper for a most thorough comparison of current methods. The new method presented in Eqs. (46–51) of that paper is most elegant, and is the fastest direct method presented there. However, as in Ref. 5, it contains an unsymmetrical transformation plus back-substitution, so that the operation count (neglecting load dependent terms) in Eq. (51), even when reduced by considering an average distribution of modifications, to

$$n \approx 3NBn_c/2 + Nn_c^2/4 \tag{32}$$

implies more than double the operations given in the present Eq. (21).

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Allen and Vincenti Blockage Corrections in a Wind Tunnel

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Introduction

In conducting wind-tunnel tests it is quite often necessary to use cylinders (or bodies) a little larger than desirable in order to attain the highest possible Reynolds number. Use of the large cylinders gives rise to wall-interference effects which, of course, influence whatever measurement is desired. There are several techniques which might be followed so that the wall-interference effects might be eliminated.

One of the most popular procedures for obtaining corrected drag forces for a single cylinder from wind-tunnel data is due to Allen and Vincenti.[1] For example, the Allen and Vincenti (A & V) procedure has been used by Bishop and Hassan,[2] Roshko,[3] and Delany and Sorensen[4] and Achenbach[7] to correct the measured drag coefficient on circular cylinders in wind-tunnel tests. In fact, Roshko states that the A & V procedure is the best correction method available and is believed to be fairly accurate at supercritical Reynolds numbers where C_d is nearly constant.

A recent investigation by the author led to the discovery that the A & V method is noticeably in error when used with relatively large-diameter circular cylinders in a wind tunnel.

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Allen and Vincenti Analysis

When a cylinder is placed in a wind tunnel, the flowfield is influenced to the extent that the unbounded plane-flow situation is no longer modeled exactly. Allen and Vincenti[1] performed an analysis which yielded the following equation [Eq. (67) in Ref. 1] to represent the actual drag coefficient,

$$C_d = C_d' \left\{1 - \frac{(2 - M^2)}{(1 - M^2)^{3/2}} \Lambda\sigma - \frac{(1 + 0.4M^2)}{(1 - M^2)^{3/2}} \Lambda\sigma - \tau C_d' \frac{(2 - M^2)(1 + 0.4M^2)}{1 - M^2}\right\} \tag{1}$$

in which Λ is a shape factor and σ and τ are geometric factors all tabulated in Ref. 1, C_d' is the measured drag coefficient, and M is the apparent upstream Mach number. To explain the terms in Eq. (1), I quote from Allen and Vincenti[1] that "... of the two correction terms involving $\Lambda\sigma$ in this equation, the first appears as a result of the change in dynamic pressure occasioned by the interference between the walls and the airfoil thickness; the second represents the effect of the pressure gradient induced by the interference between the walls and the wake. The correction term containing $\tau C_d'$ appears as a result of the change in dynamic pressure caused by the wall-wake interference." Very good agreement with drag data was obtained by Allen and Vincenti for cylindrical airfoils over a range of Mach numbers and airfoil sizes.

For the case of a circular cylinder in a wind tunnel with a negligible Mach number ($M \lesssim 0.2$), Eq. (1) reduces to

$$C_d = C_d'\{1 - 2.472(d/h)^2 - 0.5C_d'(d/h)\} \tag{2}$$

in which d is the cylinder diameter and h is the wind-tunnel width for a vertical cylinder. Equation (2) was also listed by Roshko[3] in which the coefficient 2.472 was replaced by 2.5 which certainly is acceptable.

The data of Fage[5] were used by Allen and Vincenti to obtain corrected drag coefficients in Eq. (1) for flow around a circular cylinder and an apparent computational error was made. I have calculated from Eq. (2) the corrected drag coefficients and in Fig. 1 a plot is shown of the uncorrected Fage data and the drag coefficient values as corrected using Eq. (2). Superimposed on the figure are the corrected values as contained

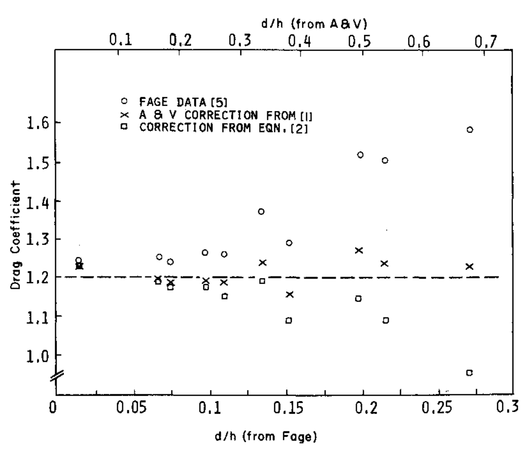

Fig. 1 Corrected and uncorrected drag coefficients as a function of blockage.